Capacitors and Capacitance

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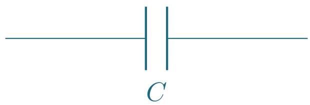
This lesson provides an introduction to an important electric-circuit element called the capacitor and the equally important concept of capacitance. When you complete this lesson, you should know the following:

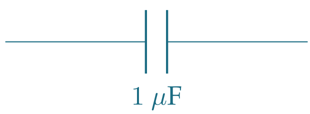
1. How capacitors are characterized by a quantity called capacitance.
2. The mathematical relationship between the voltage across and current through a capacitor.
3. The voltage-current relationship for a capacitor in a steady-state sinusoidal (AC) circuit.

Capacitors and Capacitance

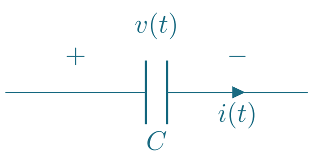
A capacitor is a common electric-circuit element that has a very special mathematical relationship between the current through and voltage across its terminals. The schematic symbol we use for a capacitor is



where the quantity C is called the *capacitance*. The unit for capacitance is called the farad, and we denote this unit with the symbol F. If, for instance, a capacitor has a capacitance equal to 1 μF, then we might represent the capacitor in a circuit like this:



If we label the voltage across and current through a capacitor like this



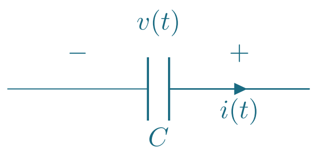
then the voltage and current are related through a mathematical relationship like this:

$$i(t) = C\frac{dv(t)}{dt}.$$

That is, the current is proportional to the derivative of the voltage with respect to time, and the constant of proportionality is equal to the capacitance. Because of this relationship, the capacitance must have units equal to ampere-seconds per volt, so a farad is equal to one ampere-second divided by one volt:

$$\mathrm{F} = \frac{\mathrm{A} \cdot \mathrm{s}}{\mathrm{V}}.$$

When relating the voltage across to the current through a capacitor, it is important to pay careful attention to the voltage polarity and current direction. If, for instance, the voltage and current are labeled like this:



then the voltage and current would be related by the following relationship:

$$i(t) = -C\frac{dv(t)}{dt}.$$

If we know the current through a capacitor, but want to determine the voltage across, then we should use the following relationship:

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau.$$

Capacitors in Steady-state AC Circuits

Suppose that the current through a capacitor is a steady-state AC function like this:

$$i(t) = A_I \cos(2\pi f t + \theta_I),$$

where f is the frequency (in Hz), A_I is the amplitude (in A) and θ_I is the phase (in radians). In this case, the voltage across the capacitor is

$$
\begin{aligned}
v(t) &= \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \\
&= \frac{1}{C}\int_{-\infty}^{t} A_I\cos(2\pi f\tau + \theta_I)d\tau \\
&= \frac{1}{C}\left[\frac{A_I}{2\pi f}\sin(2\pi f\tau + \theta_I)\right] \\
&= \frac{1}{C}\left[\frac{A_I}{2\pi f}\cos(2\pi f t + \theta_I - \pi/2)\right] \\
&= \frac{1}{2\pi f C}A_I\cos(2\pi f t + \theta_I - \pi/2).
\end{aligned}
$$

If we represent the current through the capacitor by its phasor

$$I = A_I \underline{/\theta_I},$$

then the phasor representation for the voltage across the capacitor is

$$
\begin{aligned}
V &= \frac{1}{2\pi f C}A_I\underline{/\theta_I - \pi/2} \\
&= A_I\underline{/\theta_I} \times \frac{1}{2\pi f C}\underline{/-\pi/2}.
\end{aligned}
$$

Because of this, we associate a capacitor with its *steady-state impedance*:

$$
\begin{aligned}
Z_C &= \frac{1}{2\pi f C}\underline{/-\pi/2} \\
&= \frac{1}{j2\pi f C} = -\frac{j}{2\pi f C},
\end{aligned}
$$

so that the stead-state AC relationship between the voltage across and current through a capacitor is

$$V = IZ_C.$$

This relationship tells us that the phase of a capacitor's impedance is always equal to $-90°$, and the magnitude of the impedance is inversely proportional to the frequency. In the special case when the frequency is equal to 0 Hz (DC), the impedance is equal to infinity. That is, a capacitor behaves like an open when it is in a steady-state DC circuit.